# On Vertically Global, Radially Local Models for Astrophysical Disks

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Astrophysical disks play a fundamental role in nature. They shape the evolution of planets, stars, and black holes by regulating the rate at which they acquire mass and exchange angular momentum with their environments. Unraveling the physical processes that govern these disks entails understanding the turbulent dynamics of magnetized plasma in the gravitational field of a central object. Fully global numerical simulations, while feasible, still remain very demanding. Therefore, there is great interest in devising local frameworks for studying the disk dynamics. During the last two decades, the framework known as the shearing box, has been extensively used for studying a small (in radius, azimuth, and height) volume around a fiducial location at the disk midplane. This framework is appropriate for studying barotropic disks, for which the pressure is only a function of the density and the angular frequency is independent of height. I this talk, I will introduce a new framework for developing vertically global, radially local models for non-barotropic disks, such as those with a non-trivial temperature structure, for which the angular frequency changes with height. I will illustrate the power and potential of this new framework with two prominent applications, namely the vertical shear instability and the magnetorotational instability. I will discuss the prospects of using this new framework to study a wide variety of astrophysical phenomena; including instabilities, convection, turbulent transport, as well as the structure and dynamics of disk coronae and winds, and the interstellar medium in galactic disks.

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| Subject : | : | oral |
| Topics | : | Astrophysics |
| Topics | : | Numerical simulations |

# Vertically Global, Radially Local Models for Astrophysical Disks

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# Some History on Local Models

- Hill 1878

- Spitzer & Schwarzschild (1953)
- Goldreich & Lynden-Bell (1965)

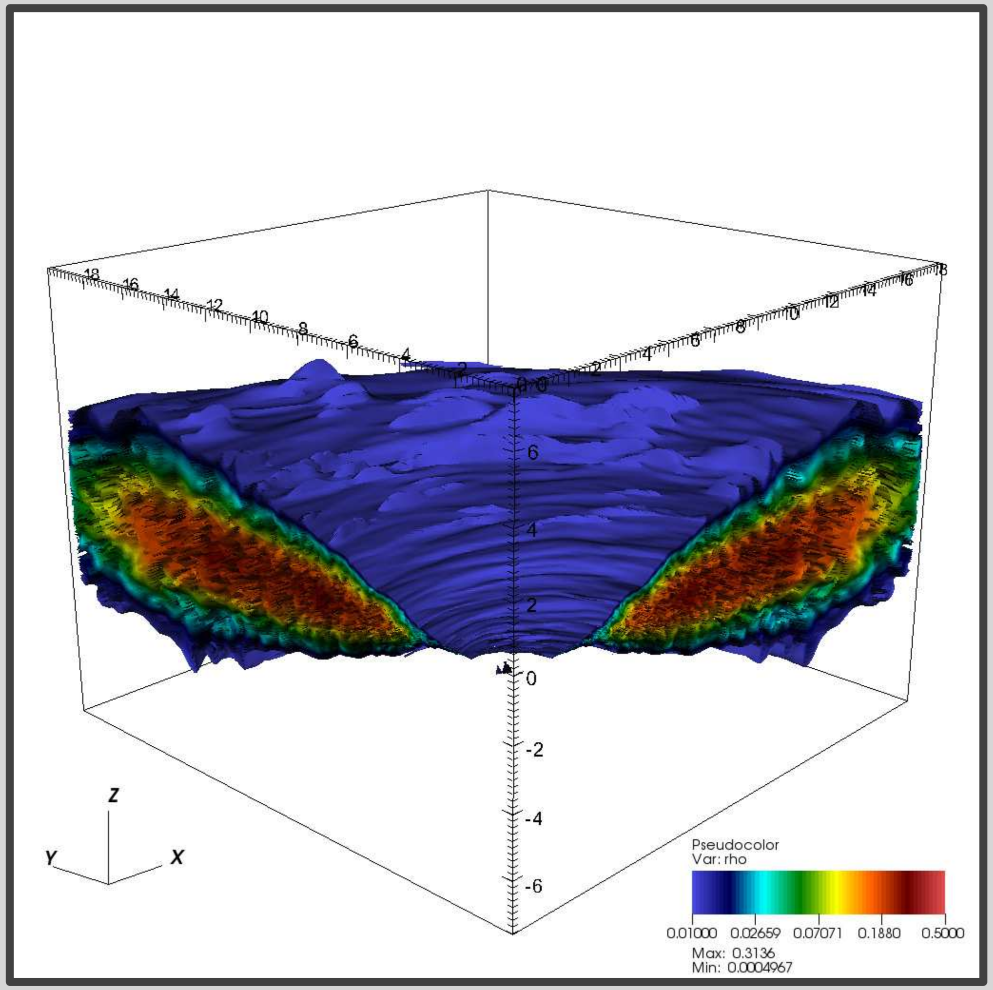
- Wisdom & Tremaine (1988)

- Hawley et al. (1995)
- Brandenburg et al. (1995)

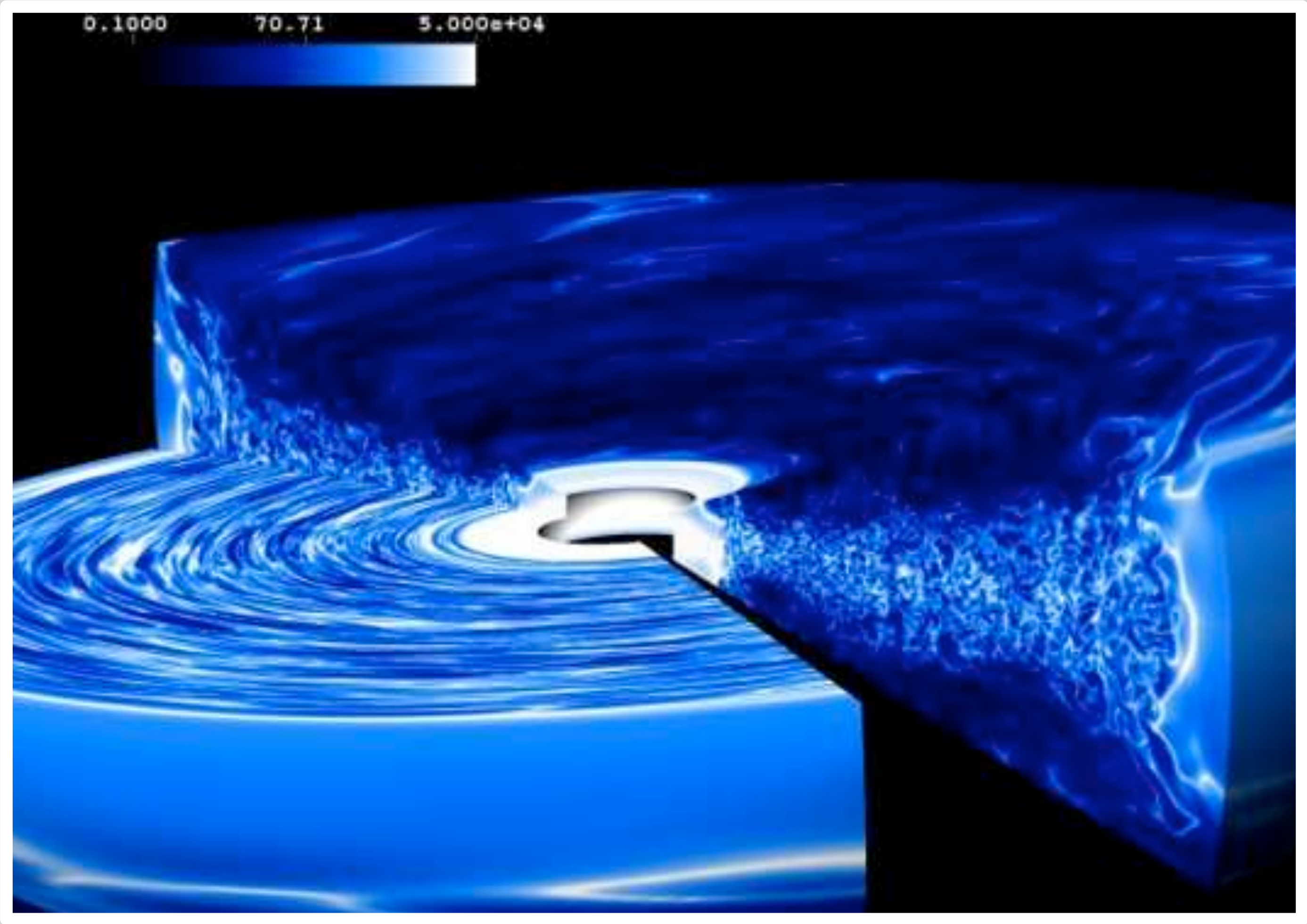
# Various Approaches

- Global prescriptions (structure but not dynamics)
- Global analytical models: 1-D, t-independent
- Local numerical simulations (box in a disk)
- Global numerical simulations (disk in a box)

# A Disk in a Box

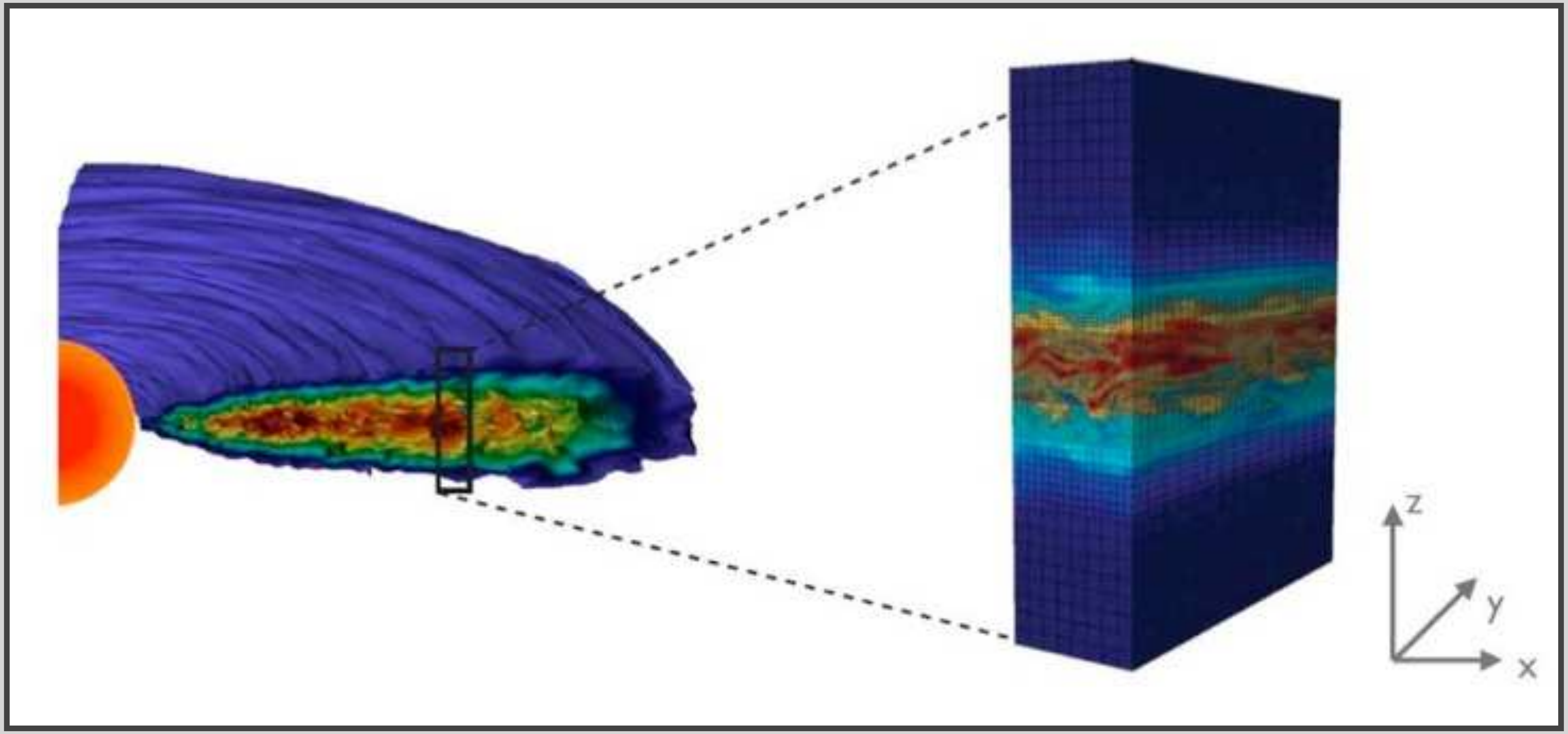



Beckwith et al. 2011




Flock et al.

# Local Models of Astrophysical Disks




Beckwith, Armitage and Simon, 2011

Essence of the local approximation: expand the equations of motion around a point corotating with the disk

# Deriving Local Disks Models

- Write down equations of motion
- Find an equilibrium solution
- Expand the equations around fiducial point
- Define boundary conditions
- Solve problems!

# Equations for an ideal MHD Fluid

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0$$

continuity

momentum

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\,\boldsymbol{v} = \Omega_{\mathrm{F}}^2 \boldsymbol{r} - 2\boldsymbol{\Omega}_{\mathrm{F}} \times \boldsymbol{v} - \nabla \Phi - \frac{\nabla P}{\rho} + \frac{1}{\rho} \boldsymbol{J} \times \boldsymbol{B}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\,\boldsymbol{B} = (\boldsymbol{B} \cdot \nabla)\,\boldsymbol{v} - \boldsymbol{B}\,(\nabla \cdot \boldsymbol{v})$$

induction

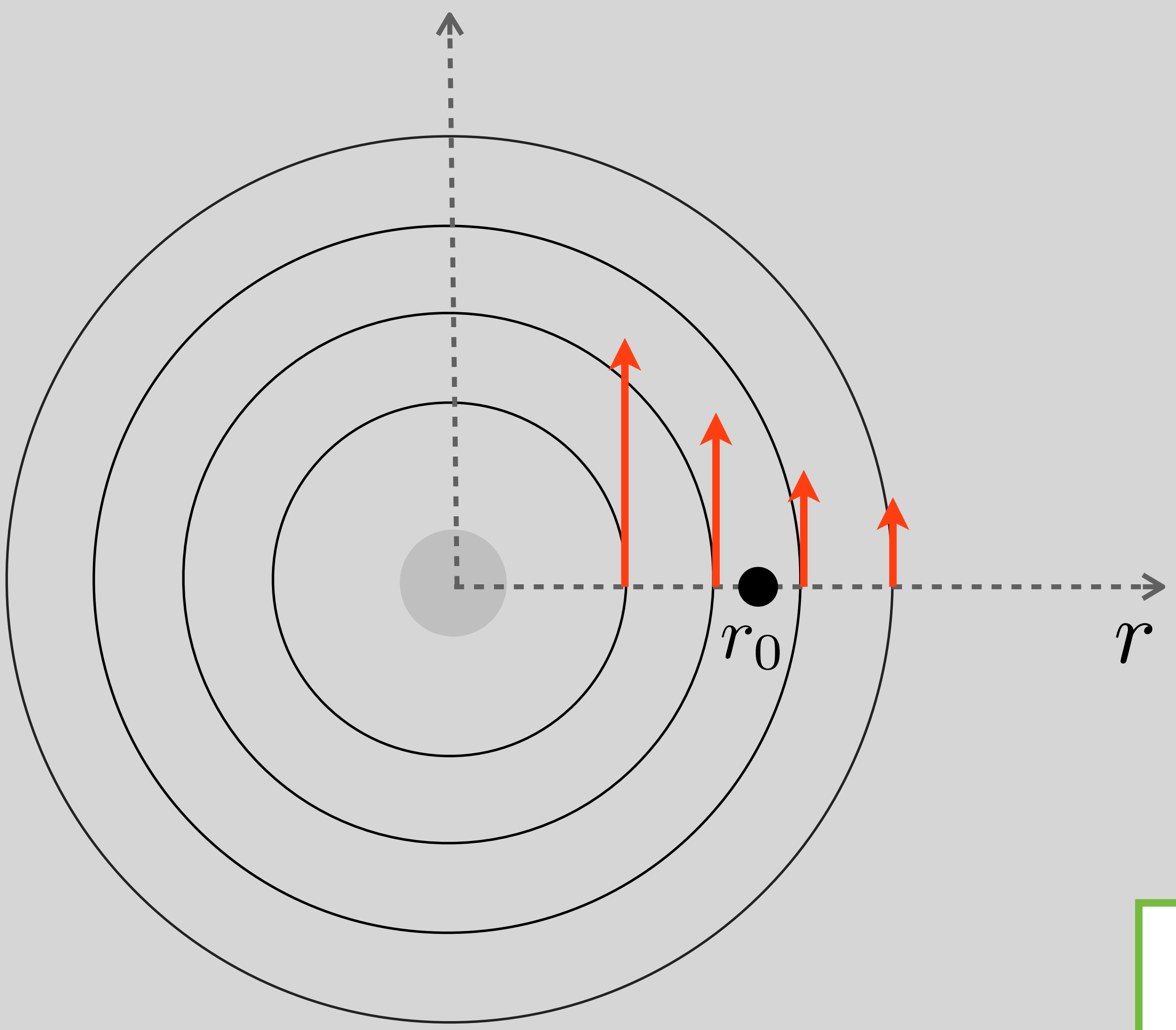
$$\frac{\partial e}{\partial t} + \nabla \cdot (e \boldsymbol{v}) = -P\,(\nabla \cdot \boldsymbol{v})$$

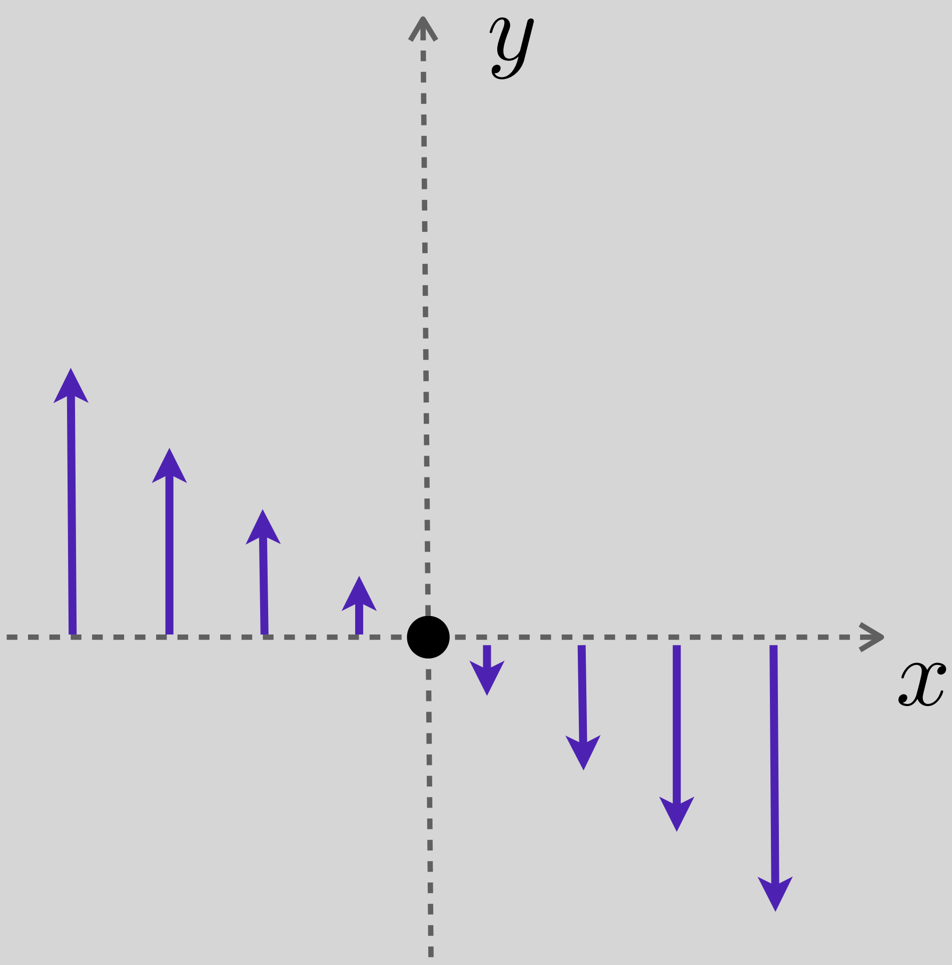
energy

# The Local Approximation

$$V(r) = r\,\Omega(r)$$

$$V(x) \simeq V_0 + S_0\,x$$




$$V(x) \simeq r_0 \left[ \Omega_0 + \left.\frac{d\Omega}{dr}\right|_0 (r - r_0) \right]$$

# Vertically Local & Radially Local

- 1. Bulk disk flow

$$\boldsymbol{V}(r) = r\Omega(r)\hat{\boldsymbol{\phi}}$$

$$\Omega^2(r) \equiv \frac{1}{r}\frac{\partial \Phi}{\partial r} = \frac{GM}{r^3}$$

$$\frac{1}{\rho_h}\frac{\partial P(\rho_h)}{\partial z} = -\frac{\partial \Phi}{\partial z}$$

- 2. Taylor expand in r

$$V(x) \equiv V_0 + S_0 x$$

$$\Omega(x) \equiv \Omega_0 + \left.\frac{\partial \Omega(r)}{\partial r}\right|_{r=r_0} x$$

$$S_0 \equiv r_0 \left.\frac{\partial \Omega(r)}{\partial r}\right|_{r=r_0}$$

- 3. Define departures

$$\boldsymbol{w} \equiv \boldsymbol{v} - V(x)\hat{\boldsymbol{y}}$$

# Standard Shearing Box - Model

$$\boldsymbol{w} \equiv \boldsymbol{v} - S_0 x \hat{\boldsymbol{y}}$$

$$\mathcal{D}_0 \equiv \partial_t + S_0 x \partial_y$$

$$\left(\mathcal{D}_0 + \boldsymbol{w} \cdot \nabla\right) \boldsymbol{w} = -2\Omega_0 \hat{\boldsymbol{z}} \times \boldsymbol{w} - S_0 w_x \hat{\boldsymbol{y}} - \frac{\nabla P}{\rho} - \frac{\partial \Phi_0(z)}{\partial z} \hat{\boldsymbol{z}} + \frac{1}{\rho} \boldsymbol{J} \times \boldsymbol{B}$$

$$\left(\mathcal{D}_0 + \boldsymbol{w} \cdot \nabla\right) \boldsymbol{B} = S_0 B_x \hat{\boldsymbol{y}} + \left(\boldsymbol{B} \cdot \nabla\right) \boldsymbol{w} - \boldsymbol{B} \left(\nabla \cdot \boldsymbol{w}\right)$$

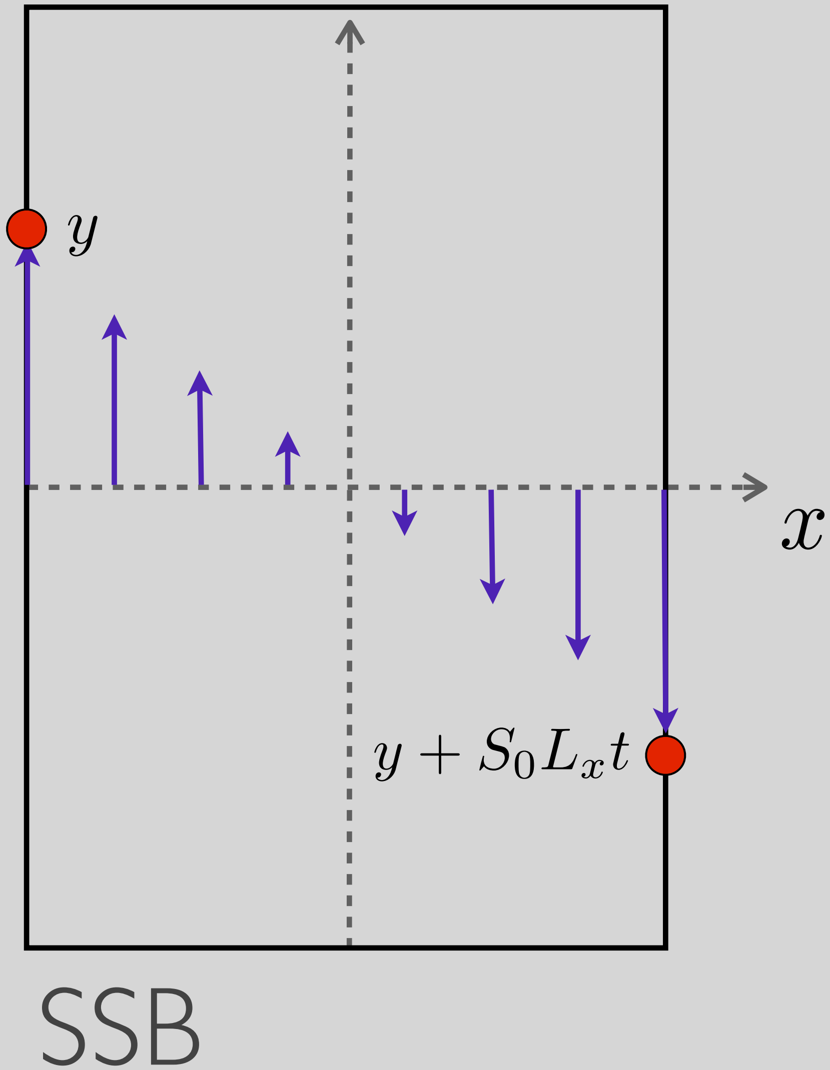
Isothermal, thin disks pressure support is negligible and angular frequency is height-independent

# Shearing Periodic Boundaries

$$\mathcal{D}_0^{\mathrm{SSB}} \equiv \partial_t + x S_0 \partial_y$$

$$y' = y - x\, S_0\, t$$

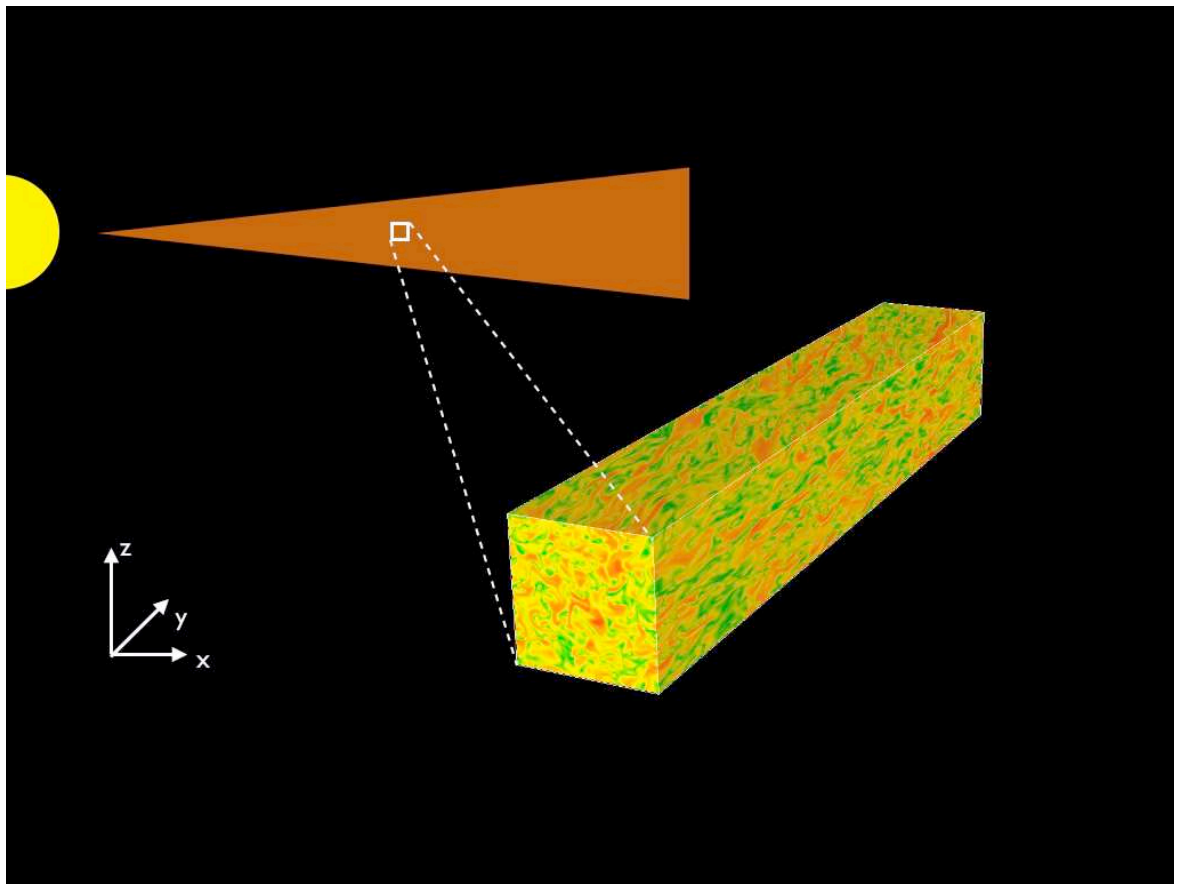
$$\mathcal{D}_0^{\mathrm{SSB}} = \partial_t'$$



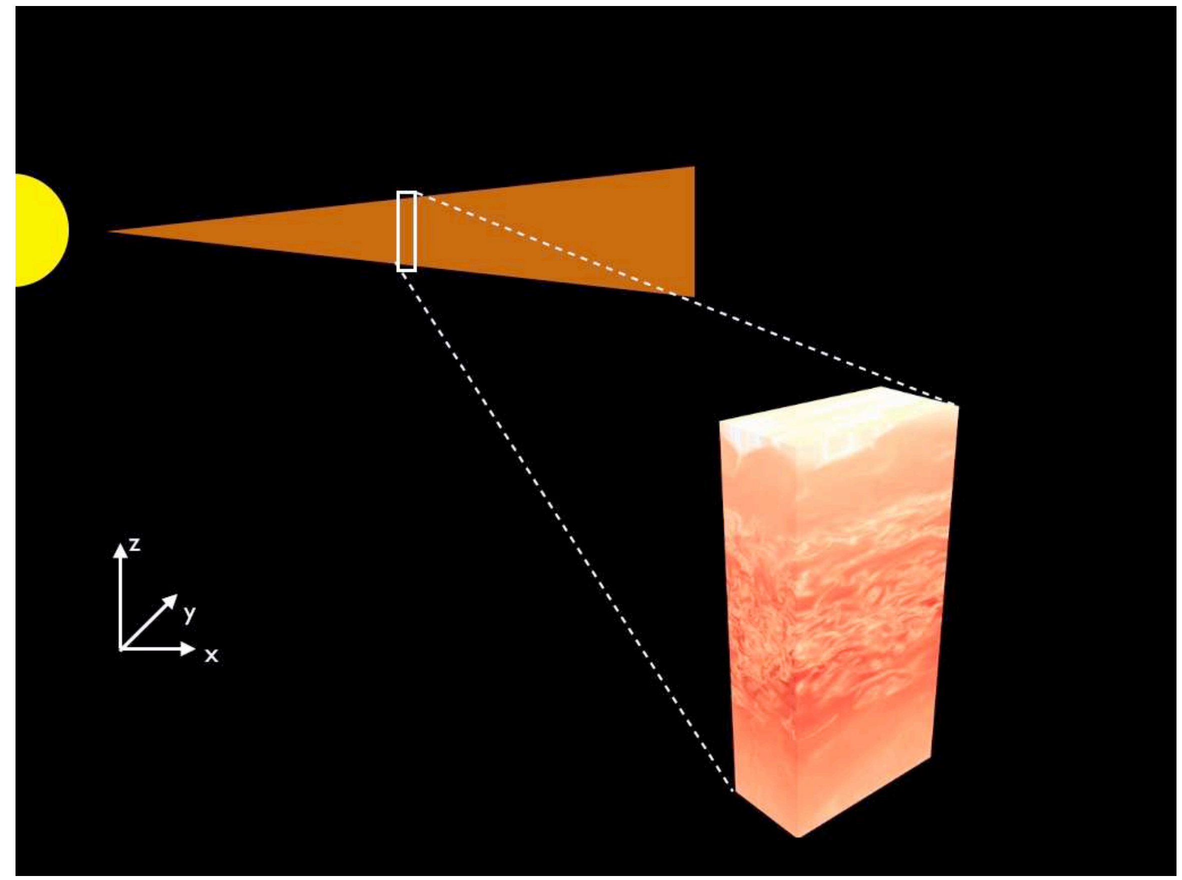

$$f(x', y', z', t') = f(x' + L_x, y', z', t')$$

$$f(x, y, z, t) = f(x + L_x, y + S_0 L_x t, z, t)$$

# The Standard Shearing Box (SSB)




$$\frac{\partial \Phi}{\partial z} \simeq 0$$




$$\frac{\partial \Phi}{\partial z} \simeq z\Omega_0^2$$

from J. Simon's website

# The Standard Shearing Box (SSB)

Pros:

- Offers a controlled environment

- Can study small-scale turbulent dynamics

Cons:

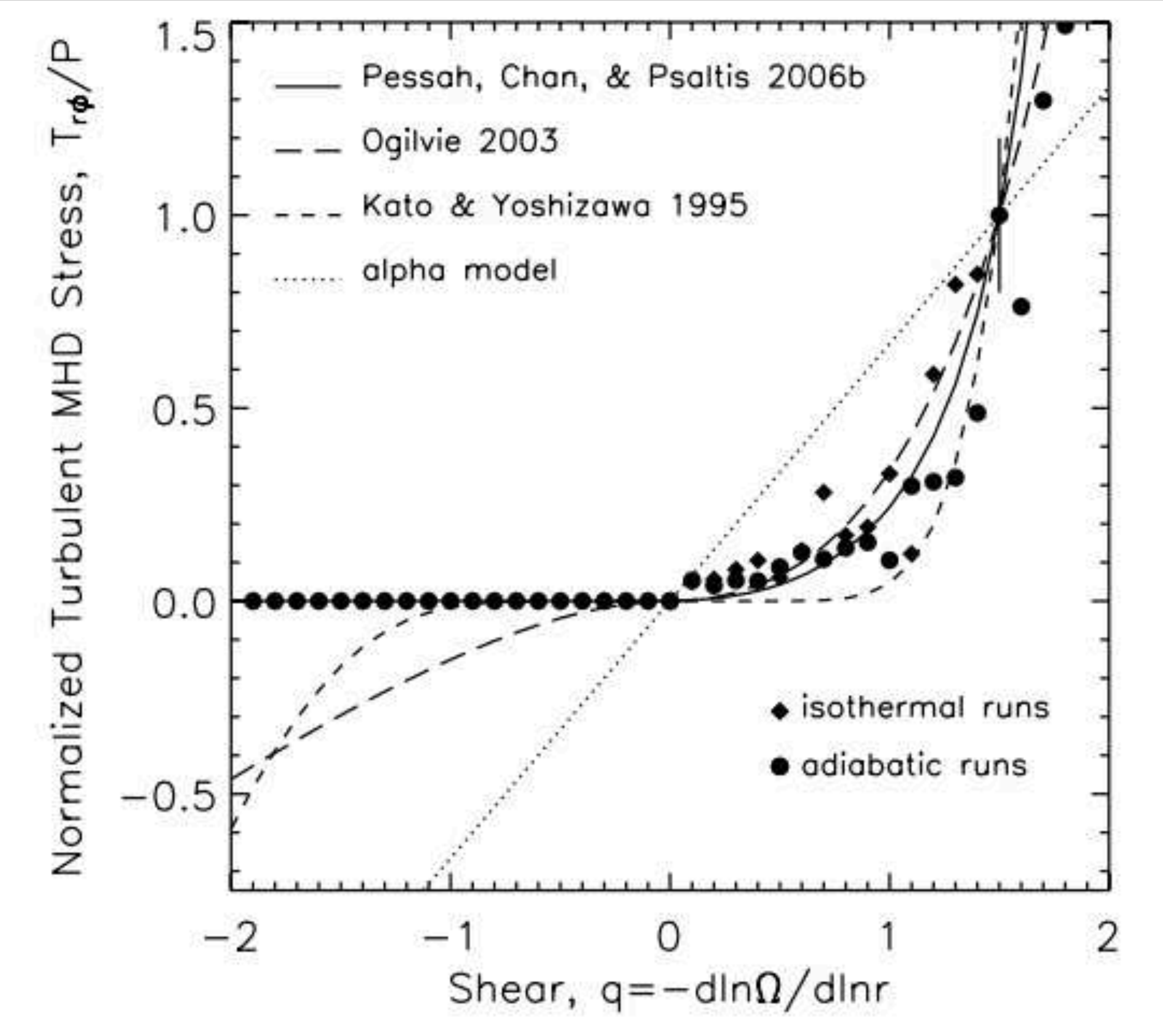
- Unable to address any global disc dynamics

- Framework valid for isothermal, thin disks

# Controlled Experiment Example

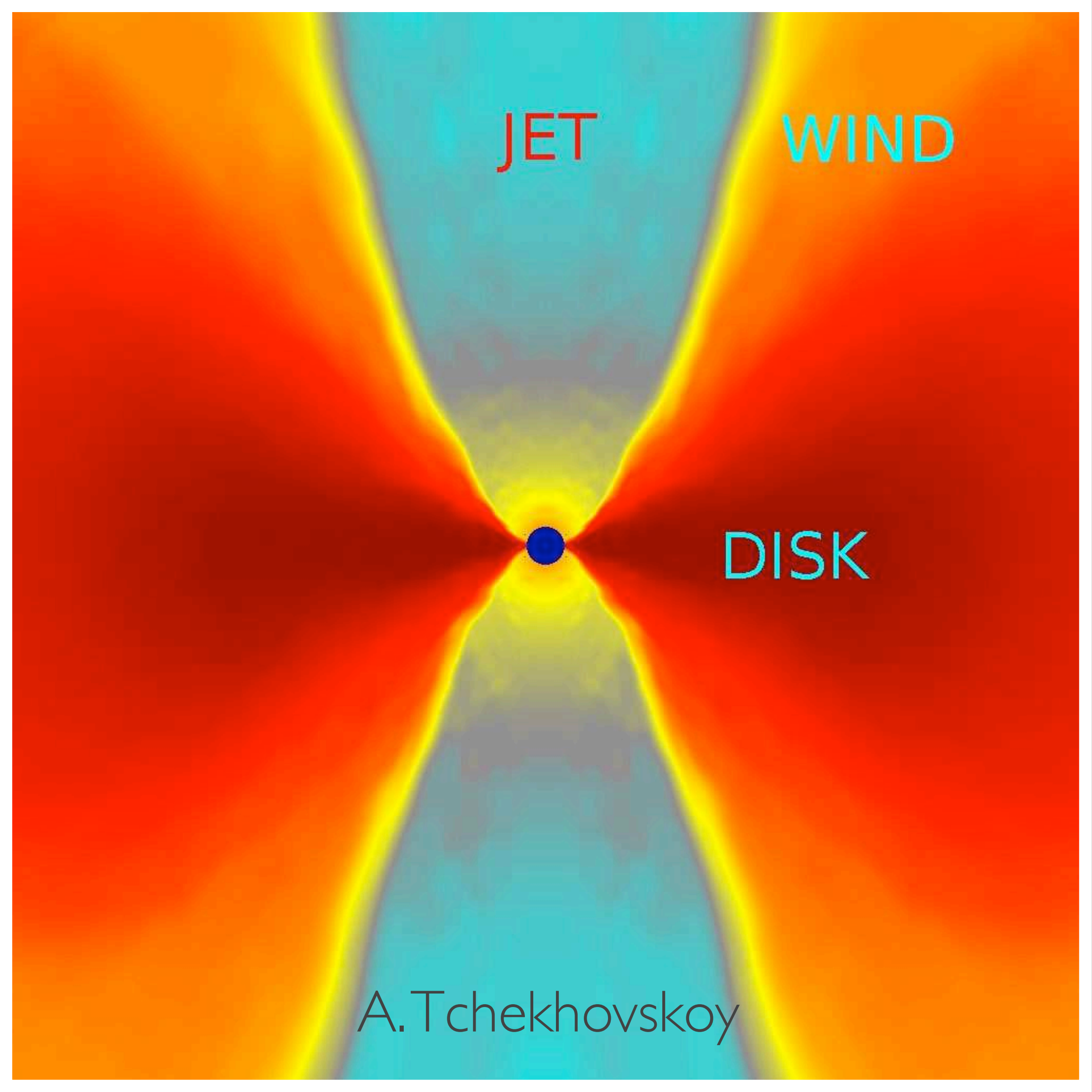
$$T_{r\phi}^{\rm ss} = q\alpha P$$

- Stress not linear in shear
- No stress for negative q




Pessah, Chan, & Psaltis 2008

# What if we care about z ~ R?


$$\Phi(r,z) = \frac{-GM}{\sqrt{r^2+z^2}}$$

$$\frac{\partial \Phi}{\partial z} = z\,\Omega_0^2\left(1+\frac{z^2}{r_0^2}\right)^{-3/2}$$

# Beyond Vertically Local Models

- Isothermal disks are barotropic $P = P(\rho)$
- Barotropic fluids rotate on cylinders $\frac{d\Omega}{dz} = 0$

- Non-trivial thermal structure implies $\frac{d\Omega}{dz} \neq 0$
- Thick disks have pressure support $\nabla P \neq 0$

# Vertically Global & Radially Local

- 1. Bulk disk flow

$$\boldsymbol{V}(r,z) = r\left[\Omega(r,z) - \Omega_{\mathrm{F}}\right]\hat{\boldsymbol{\phi}}$$

$$\Omega^2(r,z) \equiv \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r\rho_h}\frac{\partial P(\rho_h, e_h)}{\partial r}$$

$$\frac{1}{\rho_h}\frac{\partial P(\rho_h, e_h)}{\partial z} = -\frac{\partial \Phi}{\partial z}$$

- 2. Taylor expand in r

$$V(x,z) \equiv V_0(z) + S_0(z)x$$

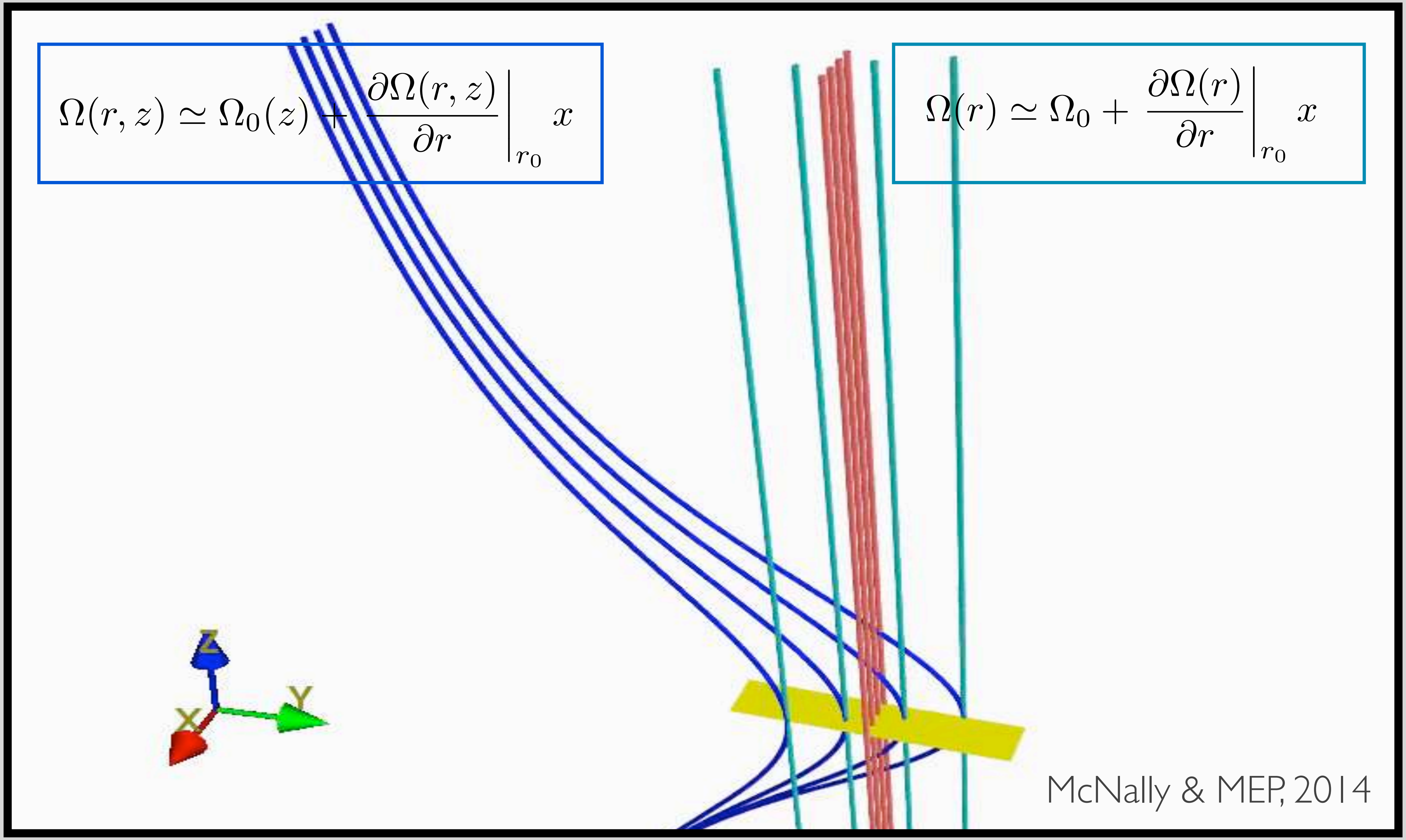
$$\Omega(x,z) \equiv \Omega_0(z) + \left.\frac{\partial \Omega(r,z)}{\partial r}\right|_{r=r_0} x$$

$$S_0(z) \equiv r_0 \left.\frac{\partial \Omega(r,z)}{\partial r}\right|_{r=r_0}$$

- 3. Define departures

$$\boldsymbol{w} \equiv \boldsymbol{v} - \left[V_0(z) + S_0(z)x\right]\hat{\boldsymbol{y}}$$

# What is New?




- *Rotation rate depends on z*
- *Shear rate depends on z*

# Vertically Global & Radially Local

- 4. Taylor-expand background equilibrium in r

$$\rho_h(r,z) = \rho_{h0}(z) + \mathrm{O}\left(\frac{x}{r_0}\right)$$

$$e_h(r,z) = e_{h0}(z) + \mathrm{O}\left(\frac{x}{r_0}\right)$$

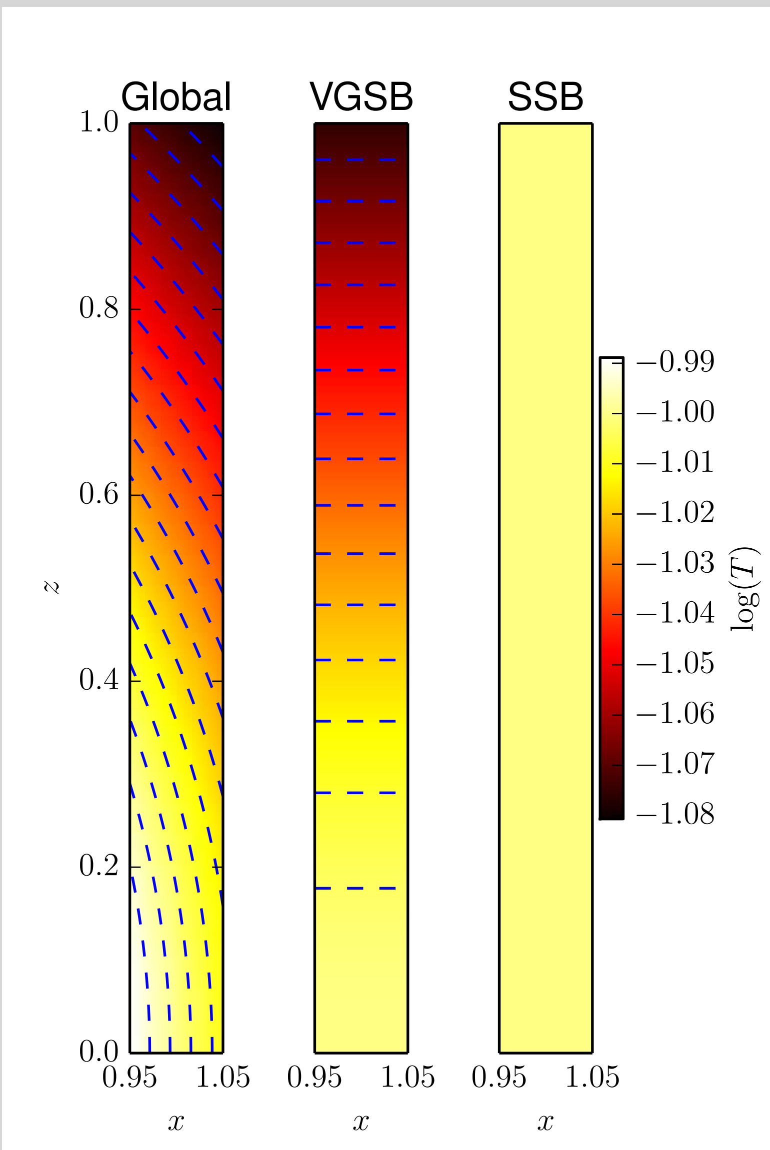
$$\frac{\nabla P(\rho_h, e_h)}{\rho_h} = \frac{\nabla P(\rho_{h0}, e_{h0})}{\rho_{h0}} + \mathrm{O}\left(\frac{x}{r_0}\right)$$

$$\frac{1}{\rho_{h0}}\frac{\partial P(\rho_{h0}, e_{h0})}{\partial z} = -\frac{\partial \Phi_0(z)}{\partial z}$$

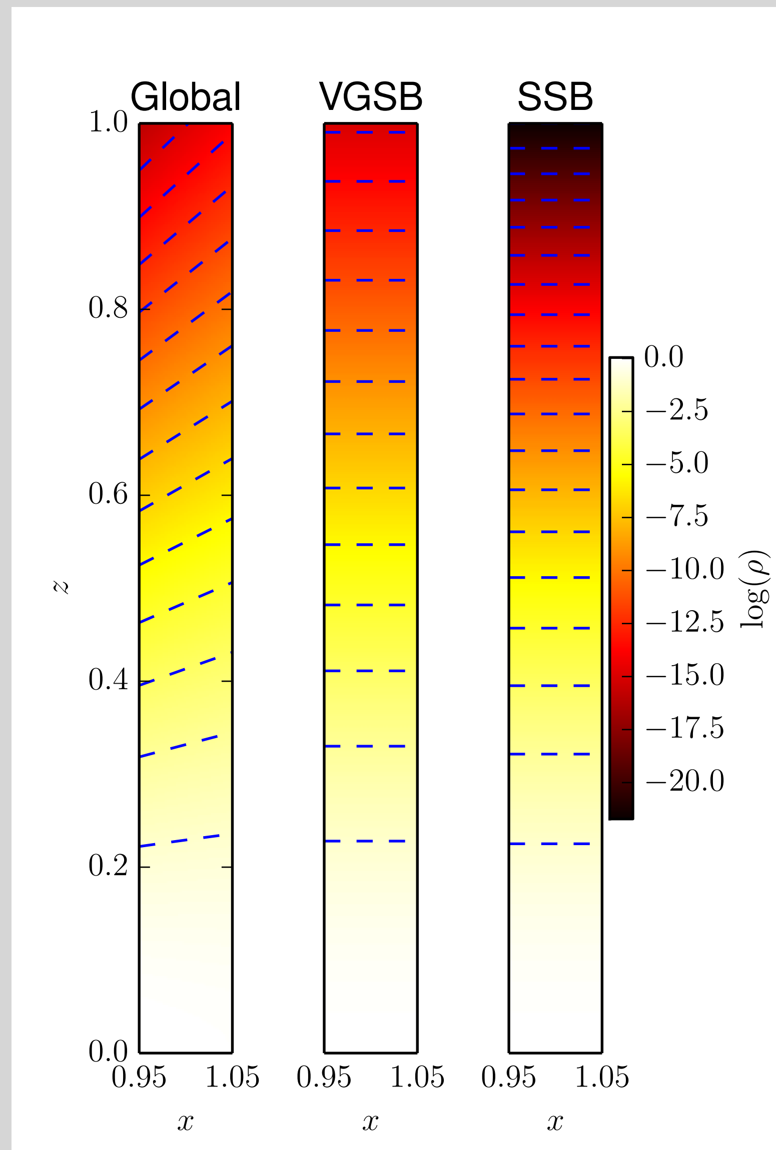
This eliminates radial gradients, which is necessary for domains with shearing-periodic boundaries

# Vertically Global & Radially Local

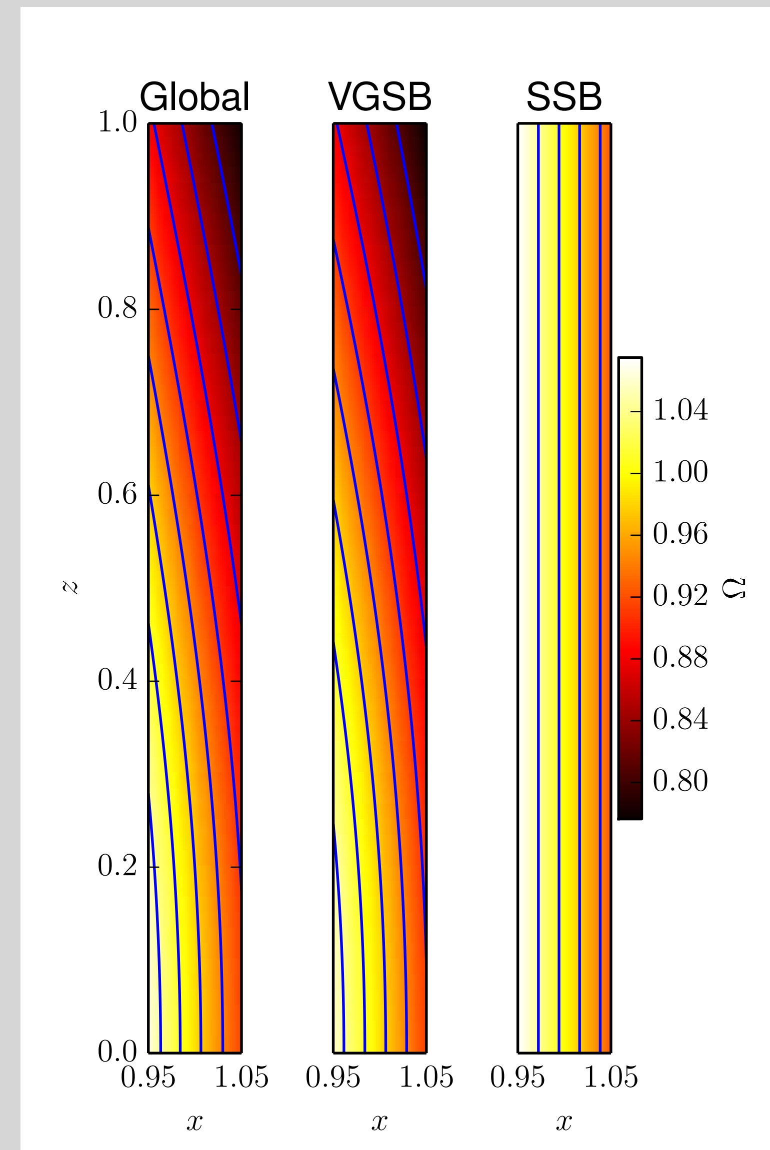
## Temperature




## Density




## Angular Frequency




McNally & MEP, 2014

# Vertically Global, Radially Local

$$\boldsymbol{w} \equiv \boldsymbol{v} - [V_0(z) + S_0(z)x]\,\hat{\boldsymbol{y}}$$

$$\mathcal{D}_0 \equiv \partial_t + [V_0(z) + S_0(z)x]\,\partial_y$$

momentum

$$(\mathcal{D}_0 + \boldsymbol{w}\cdot\nabla)\,\boldsymbol{w} + w_z \frac{\partial V(x,z)}{\partial z}\hat{\boldsymbol{y}} =$$
$$-2\Omega_0(z)\hat{\boldsymbol{z}}\times\boldsymbol{w} - S_0(z)w_x\hat{\boldsymbol{y}} - \frac{\nabla P}{\rho} - \frac{\partial \Phi_0(z)}{\partial z}\hat{\boldsymbol{z}} + \frac{1}{\rho}\boldsymbol{J}\times\boldsymbol{B}$$

induction

$$(\mathcal{D}_0 + \boldsymbol{w}\cdot\nabla)\,\boldsymbol{B} - B_z \frac{\partial V(x,z)}{\partial z}\hat{\boldsymbol{y}} =$$
$$S_0(z)B_x\hat{\boldsymbol{y}} + (\boldsymbol{B}\cdot\nabla)\,\boldsymbol{w} - \boldsymbol{B}\,(\nabla\cdot\boldsymbol{w})$$

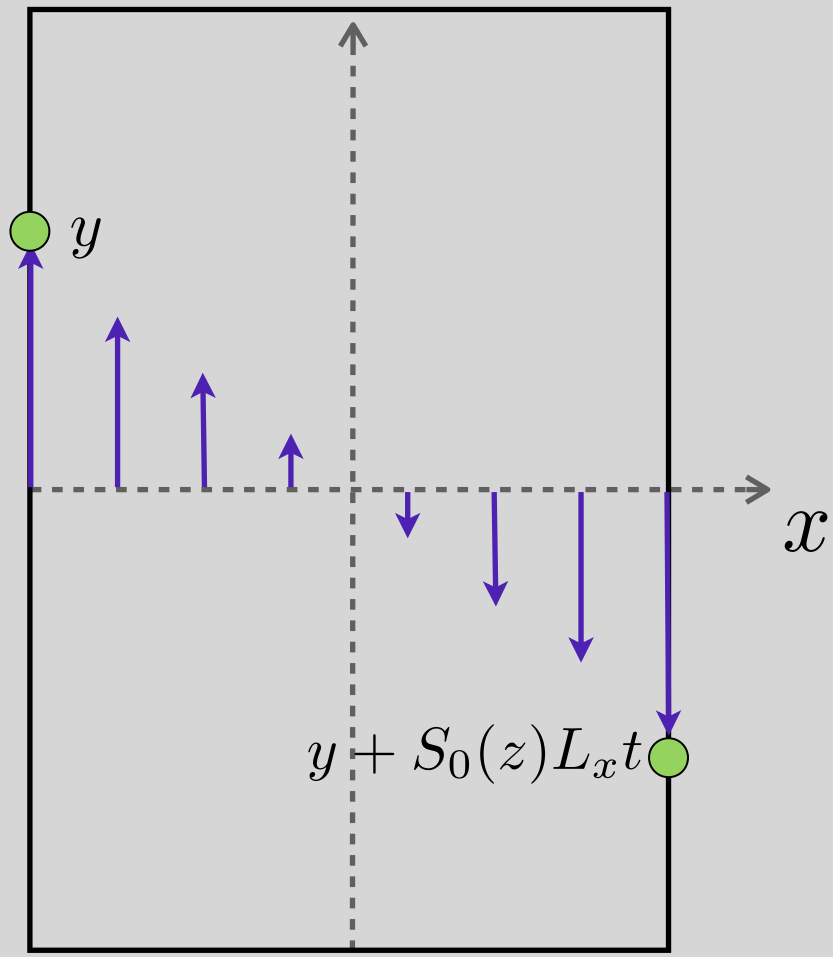
Not-consistent with shearing-periodic boundaries!

# Shearing Periodic Boundaries(z)

$$\mathcal{D}_0 \equiv \partial_t + [V_0(z) + S_0(z)x]\,\partial_y$$

$$y' = y - [V_0(z) + S_0(z)\,x]\,t$$

$$\mathcal{D}_0 = \partial_t'$$




VGSB

$$f(x', y', z', t') = f(x' + L_x, y', z', t')$$

$$f(x, y, z, t) = f(x + L_x, y + S_0(z)L_x t, z, t)$$

For this to work we only need the approx.

$$\partial_z V(x, z) = \partial_z V_0(z) + x\,\partial_z S_0(z)$$

$$\partial_z V(x, z) \simeq \partial_z V_0(z)$$

# Vertically Global Shearing Box

$$\boldsymbol{w} \equiv \boldsymbol{v} - [V_0(z) + S_0(z)x]\,\hat{\boldsymbol{y}}$$

$$\mathcal{D}_0 \equiv \partial_t + [V_0(z) + S_0(z)x]\,\partial_y$$

momentum

$$(\mathcal{D}_0 + \boldsymbol{w}\cdot\nabla)\,\boldsymbol{w} + w_z \frac{\partial V_0(z)}{\partial z}\hat{\boldsymbol{y}} = - 2\Omega_0(z)\hat{\boldsymbol{z}}\times\boldsymbol{w} - S_0(z)w_x\hat{\boldsymbol{y}} - \frac{\nabla P}{\rho} - \frac{\partial \Phi_0(z)}{\partial z}\hat{\boldsymbol{z}} + \frac{1}{\rho}\boldsymbol{J}\times\boldsymbol{B}$$

induction

$$(\mathcal{D}_0 + \boldsymbol{w}\cdot\nabla)\,\boldsymbol{B} - B_z \frac{\partial V_0(z)}{\partial z}\hat{\boldsymbol{y}} = S_0(z)B_x\hat{\boldsymbol{y}} + (\boldsymbol{B}\cdot\nabla)\,\boldsymbol{w} - \boldsymbol{B}\,(\nabla\cdot\boldsymbol{w})$$

Amenable to shearing-periodic boundary conditions !!

# Conserved Fluid Properties

- Kelvin's circulation theorem

$$\partial_t(\nabla \times \boldsymbol{v}) = \nabla \times [\boldsymbol{v} \times (\nabla \times \boldsymbol{v})]$$

$$\Gamma = \oint \boldsymbol{v} \cdot d\boldsymbol{l}$$

↓

$$[\partial_t + \boldsymbol{v} \cdot \nabla](\nabla \cdot (\nabla \times \boldsymbol{v})) = 0$$

→

$$[\partial_t + \boldsymbol{v} \cdot \nabla]\,\Gamma = 0$$

- Alfven's frozen-in theorem

$$\partial_t \boldsymbol{B} = \nabla \times [\boldsymbol{v} \times (\nabla \times \boldsymbol{B})]$$

$$\Phi_{\mathrm{B}} = \int \boldsymbol{B} \cdot d\boldsymbol{S}$$

↓

$$[\partial_t + \boldsymbol{v} \cdot \nabla](\nabla \cdot \boldsymbol{B}) = 0$$

→

$$[\partial_t + \boldsymbol{v} \cdot \nabla]\,\Phi_{\mathrm{B}} = 0$$

# Conserved Fluid Properties in VGSB

- Kelvin's circulation theorem

$$(\mathcal{D}_0 + \boldsymbol{w} \cdot \nabla)\,\Gamma = \int_{\mathrm{S}} \left[ \nabla \times \left( x w_z \frac{\partial S_0(z)}{\partial z} \hat{\boldsymbol{y}} \right) \right] + \ldots$$

- Alfven's frozen-in theorem

$$(\mathcal{D}_0 + \boldsymbol{w} \cdot \nabla)\,(\nabla \cdot \boldsymbol{B}) = -x \frac{\partial S_0(z)}{\partial z} \frac{\partial B_z}{\partial y}$$

- If barotropic or axisymmetric

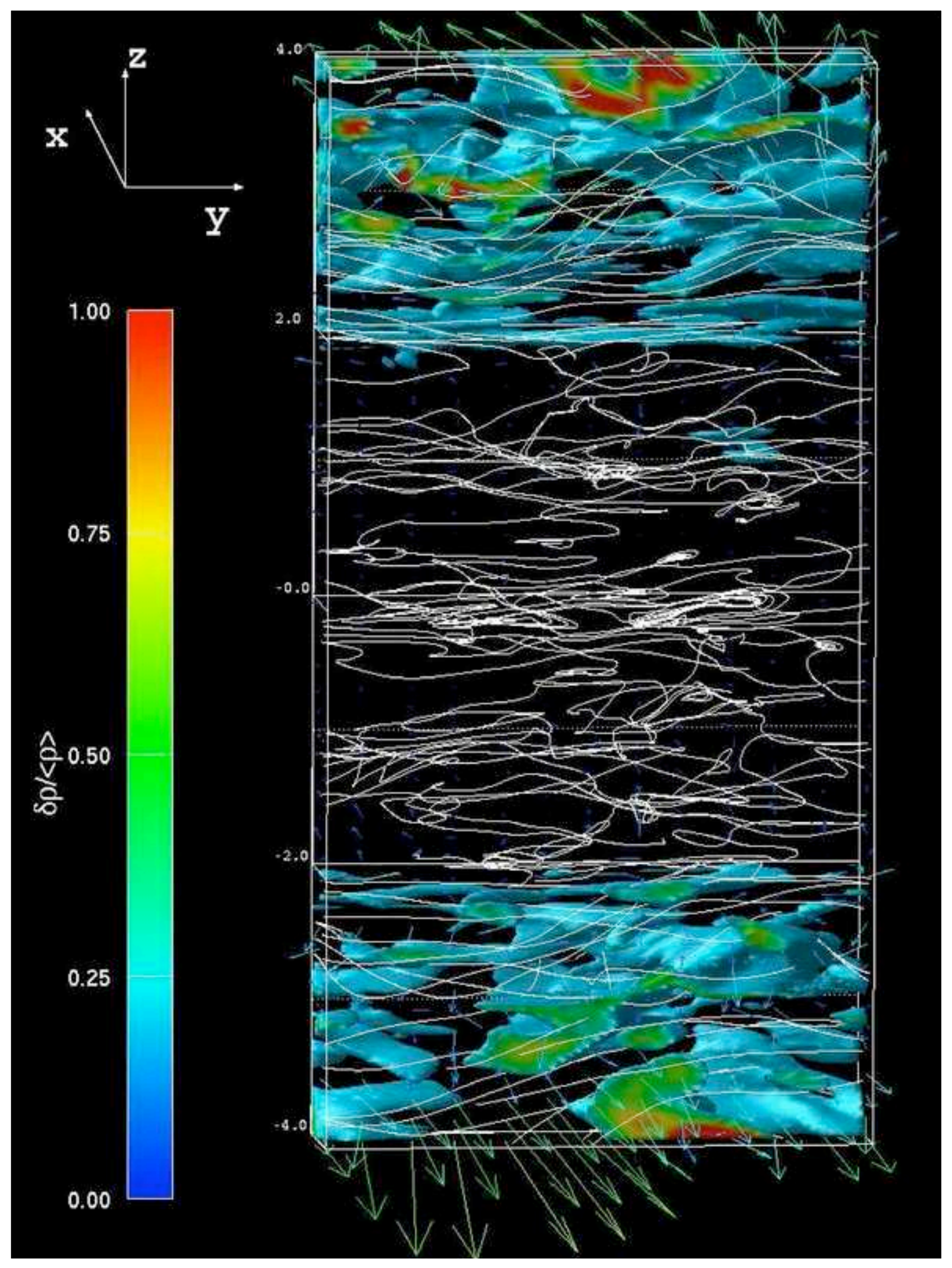
$$(\mathcal{D}_0 + \boldsymbol{w} \cdot \nabla)\,\Gamma = 0$$

$$(\mathcal{D}_0 + \boldsymbol{w} \cdot \nabla)\,\Phi_{\mathrm{B}} = 0$$

# When Will It Matter?

- Hydrodynamic Disk Instabilities
- Disk Convection
- Disk Coronae and Thick Disks
- Disk Winds
- Interstellar Medium and Galactic Disks

# Disk Winds




Suzuki & Inutsuka 2009

Usual assumption

$$\frac{\partial \Phi}{\partial z} \simeq z\Omega_0^2$$

makes it hard to launch winds

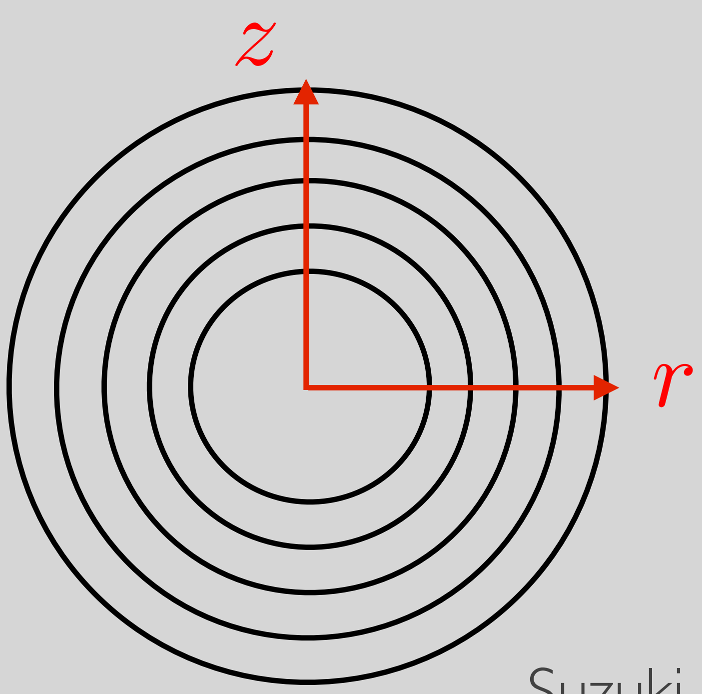
Some works have used

$$\frac{\partial \Phi}{\partial z} = z\,\Omega_0^2 \left(1 + \frac{z^2}{r_0^2}\right)^{-3/2}$$

without considering changes in rotation rate

# Spherical Temperature Disk Structure

$$T(r,z) \equiv T_0 \frac{r_0}{\sqrt{r^2+z^2}}$$




Suzuki & Inutsuka, 2013

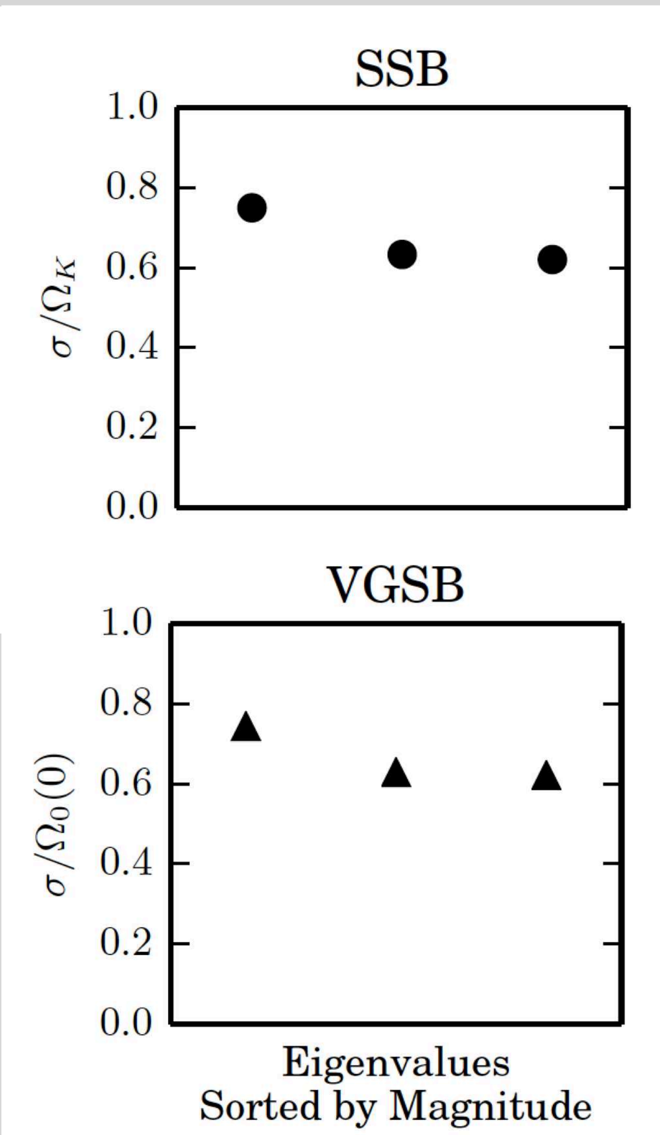
$$\rho(r,z) = \rho_0 \left(\frac{r}{\sqrt{r^2+z^2}}\right)^{\nu} \left(\frac{\sqrt{r^2+z^2}}{r_0}\right)^{1-\mu}$$

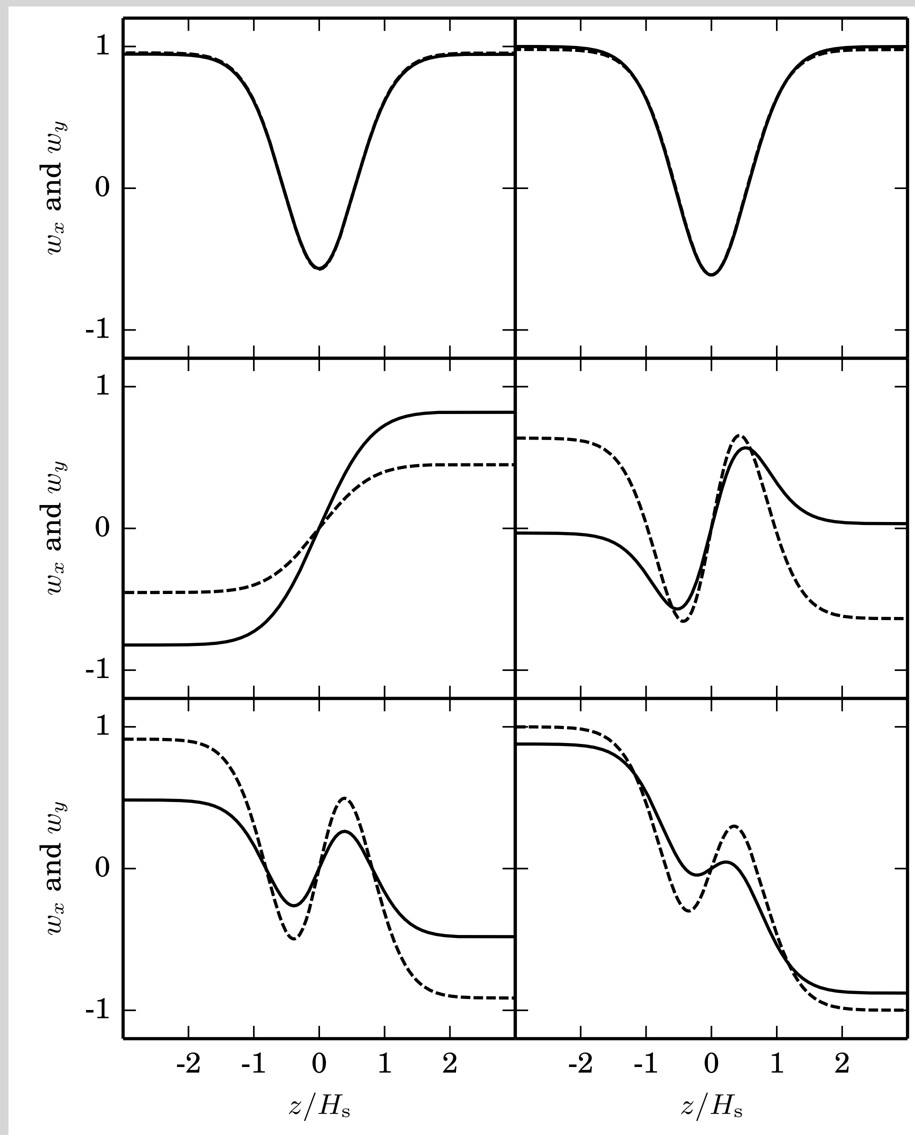
$$\Omega(r,z) = \sqrt{\nu}\,\frac{c_{\mathrm{s0}}}{r} \left(\frac{\sqrt{r^2+z^2}}{r_0}\right)^{-1/2}$$

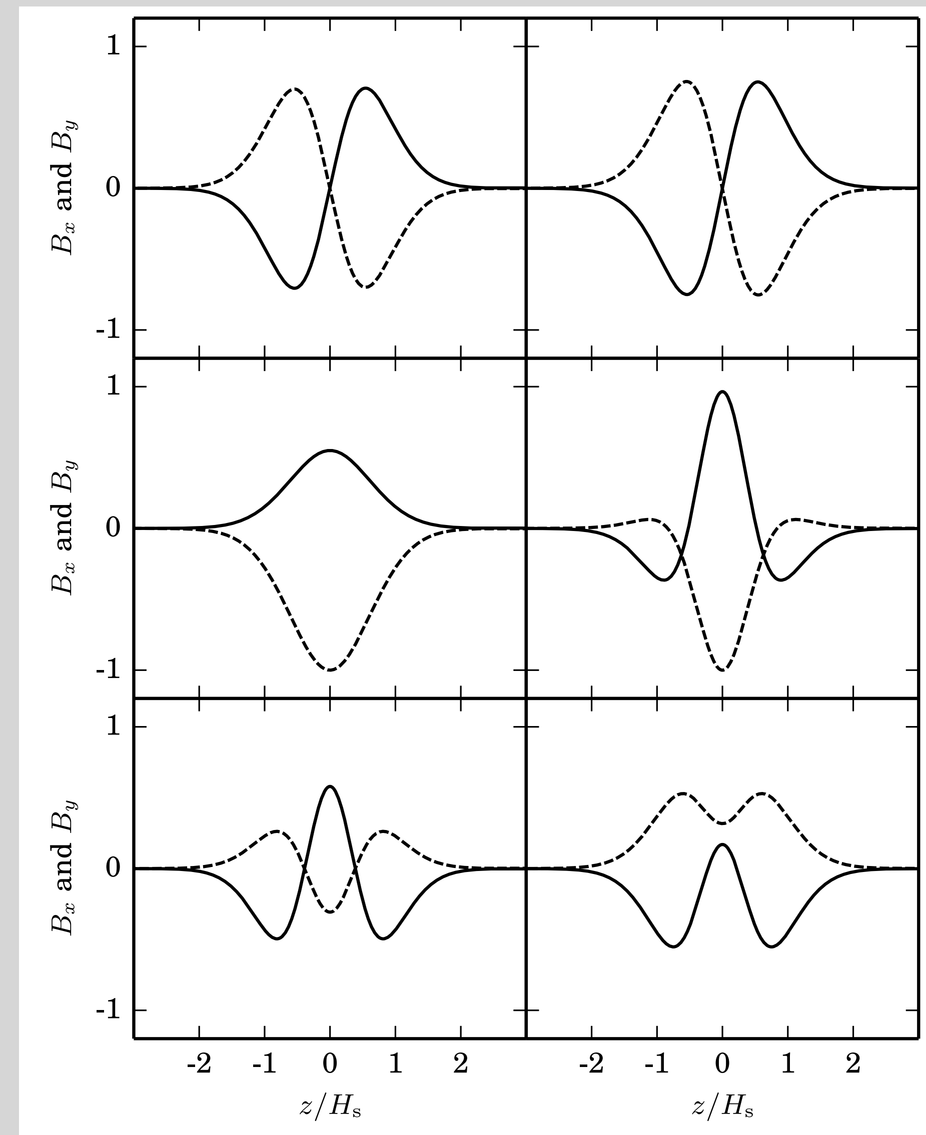
$$\nu + \mu = \frac{v_{\mathrm{K0}}^2}{c_{\mathrm{s0}}^2}$$

# Magnetorotational Instability (MRI)

SSB VGSB SSB VGSB



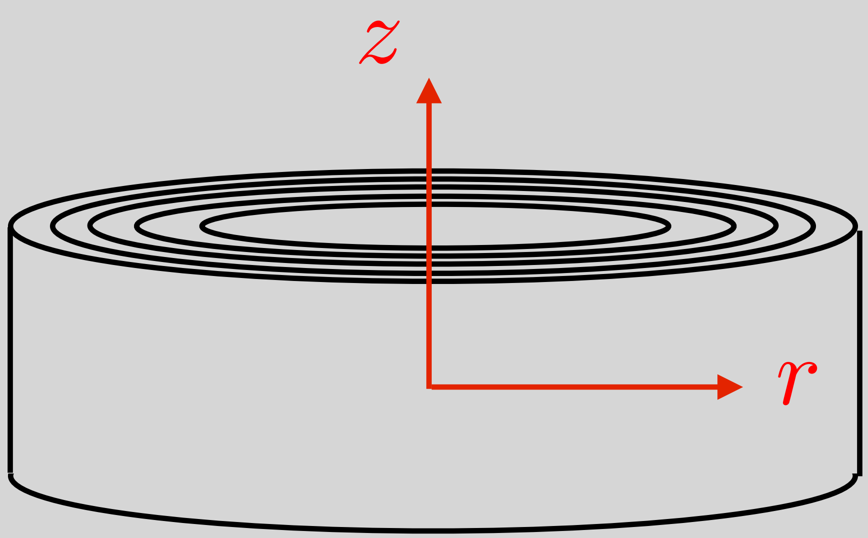

Growth Rates Velocity Field Magnetic Field

MRI only studied in isothermal disks so far!

McNally & MEP, 2014

# Cylindrical Temperature Disk Structure

$$T(r) \equiv T_0 \left(\frac{r}{r_0}\right)^q$$



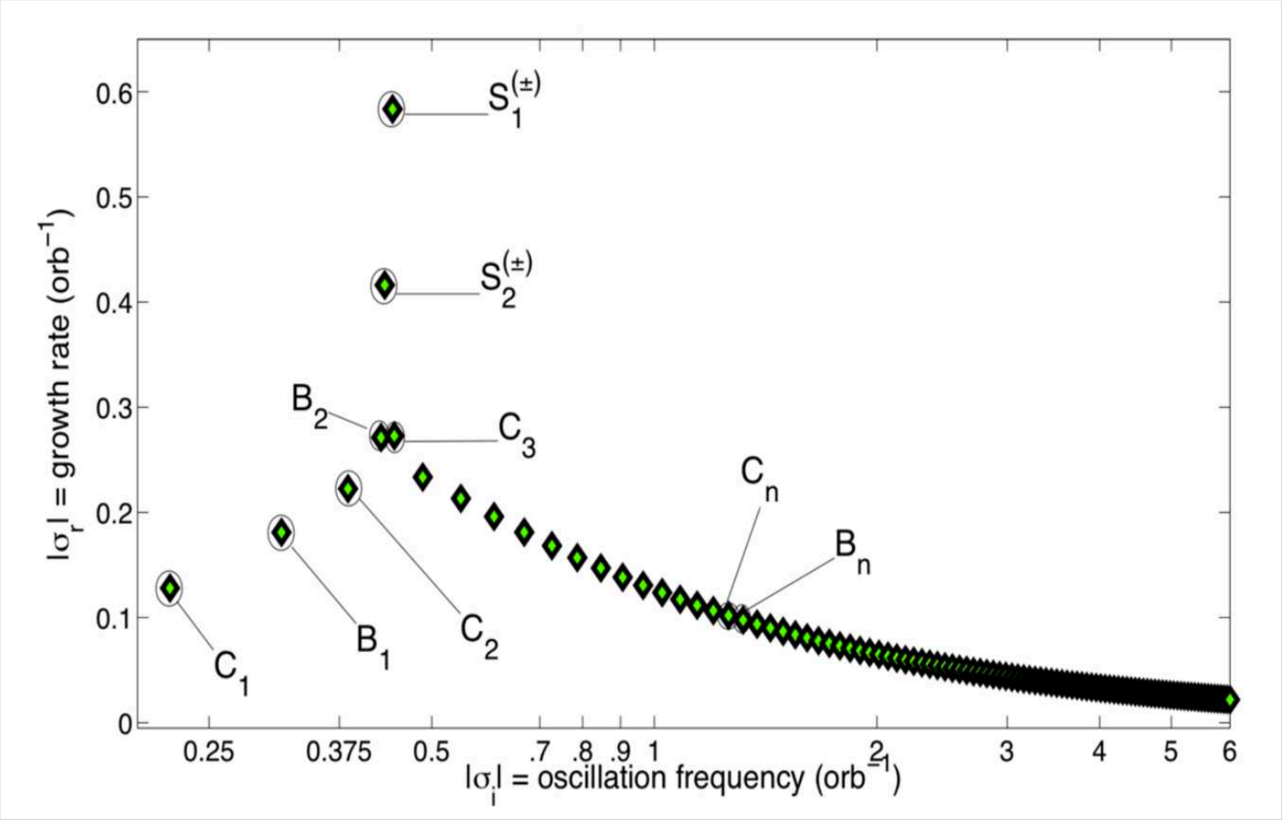

Nelson, Gressel, & Umurhan 2013

$$\rho(r,z) = \rho_0 \left(\frac{r}{r_0}\right)^p \exp\left[-\frac{v_\mathrm{K}^2}{c_\mathrm{s}^2}\left(1 - \frac{1}{\sqrt{1+(z/r)^2}}\right)\right]$$

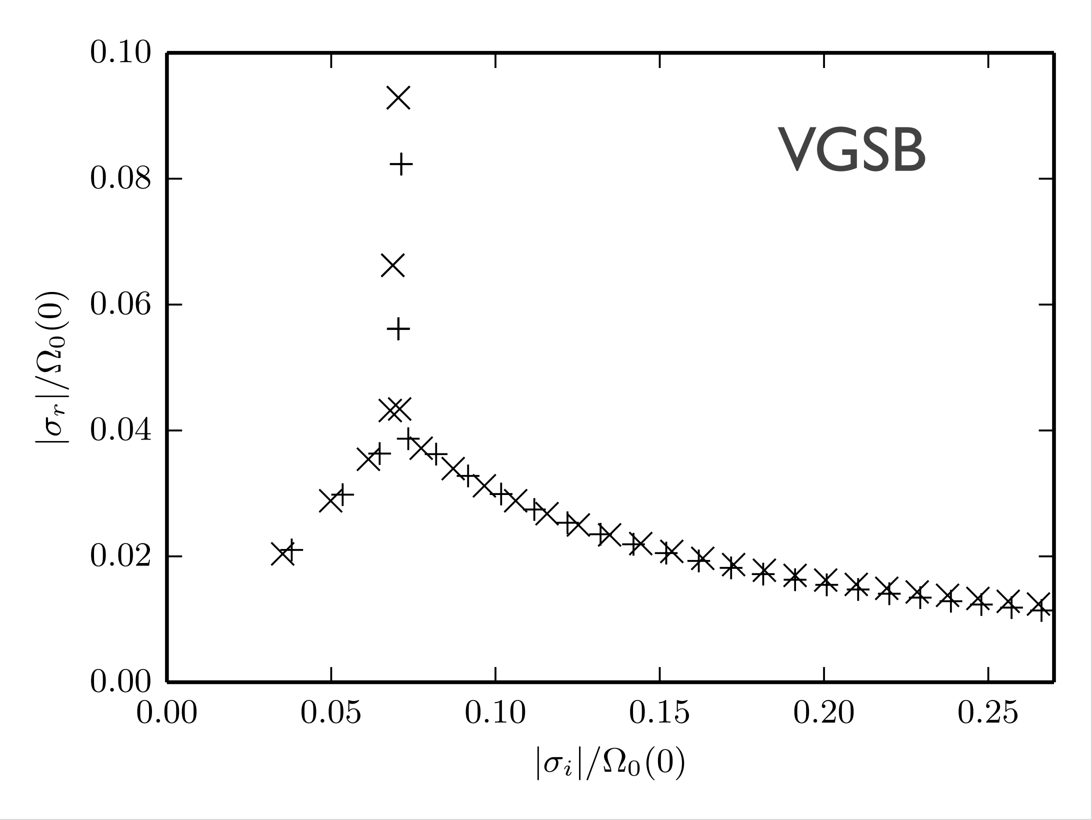
$$\Omega(r,z) = \Omega_\mathrm{K}\sqrt{1 + (p+q)\frac{c_\mathrm{s}^2}{v_\mathrm{K}^2} + q\left(1 - \frac{1}{\sqrt{1+(z/r)^2}}\right)}$$

# Vertical Shear Instability (VSI)

- Thin, isothermal hydro disks seem to be stable, but there are several instabilities that feed off vertical shear (GSF in 60's!)
- These could be important in low-ionization protoplanetary disks

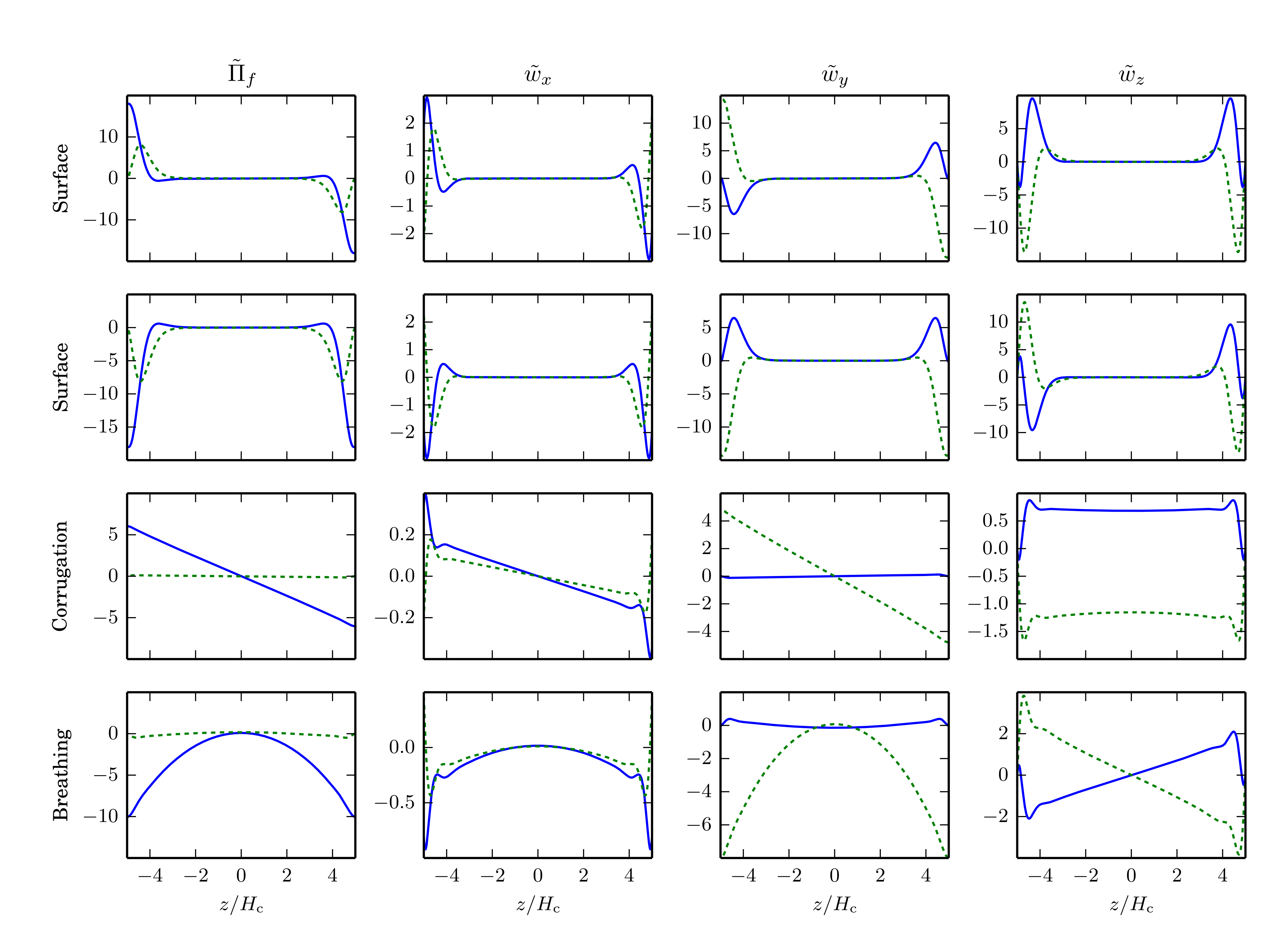



Nelson, Gressel, & Umurhan 2013




McNally & MEP, 2014

# Unstable Modes In Hydro Disks

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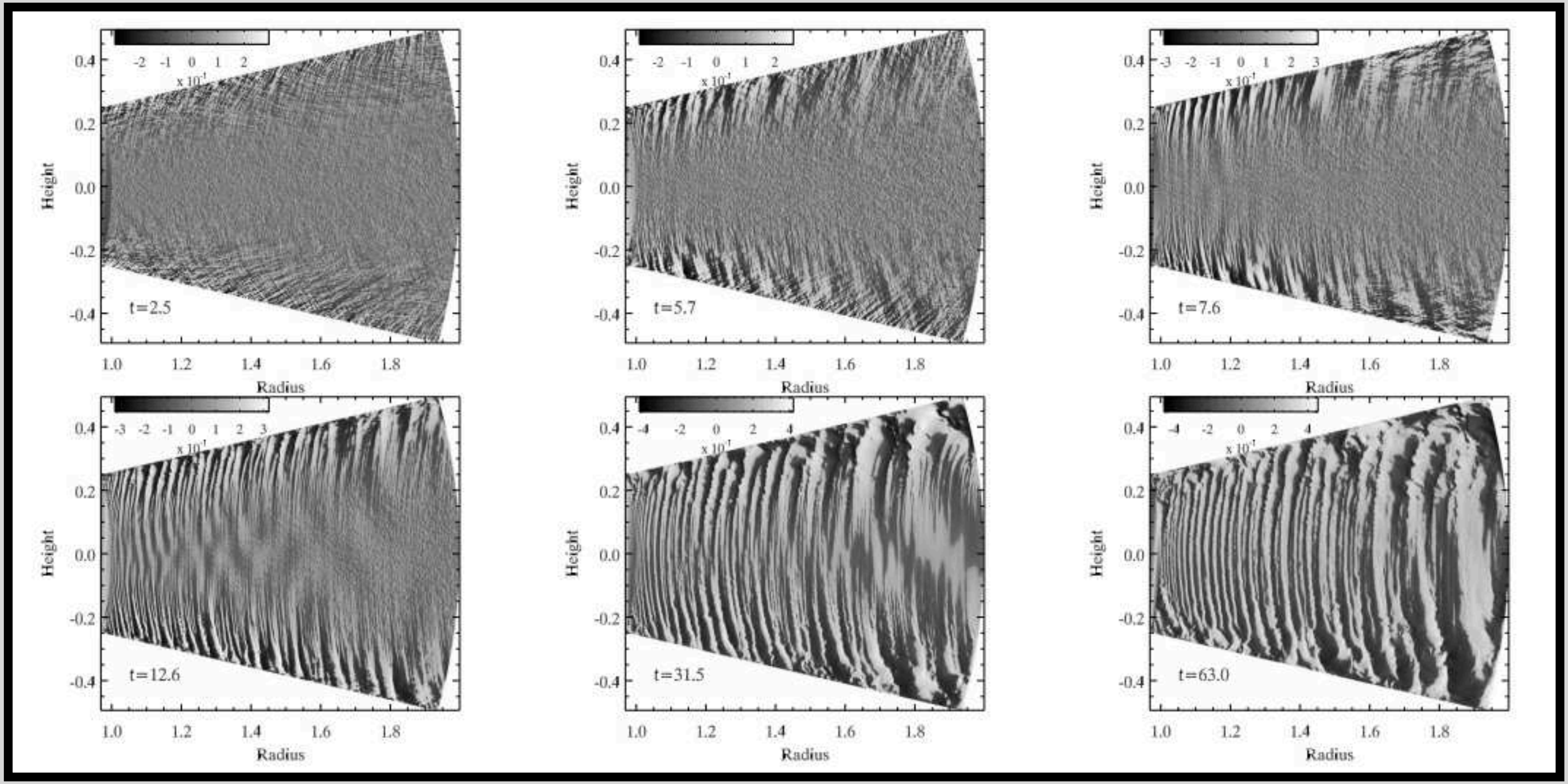
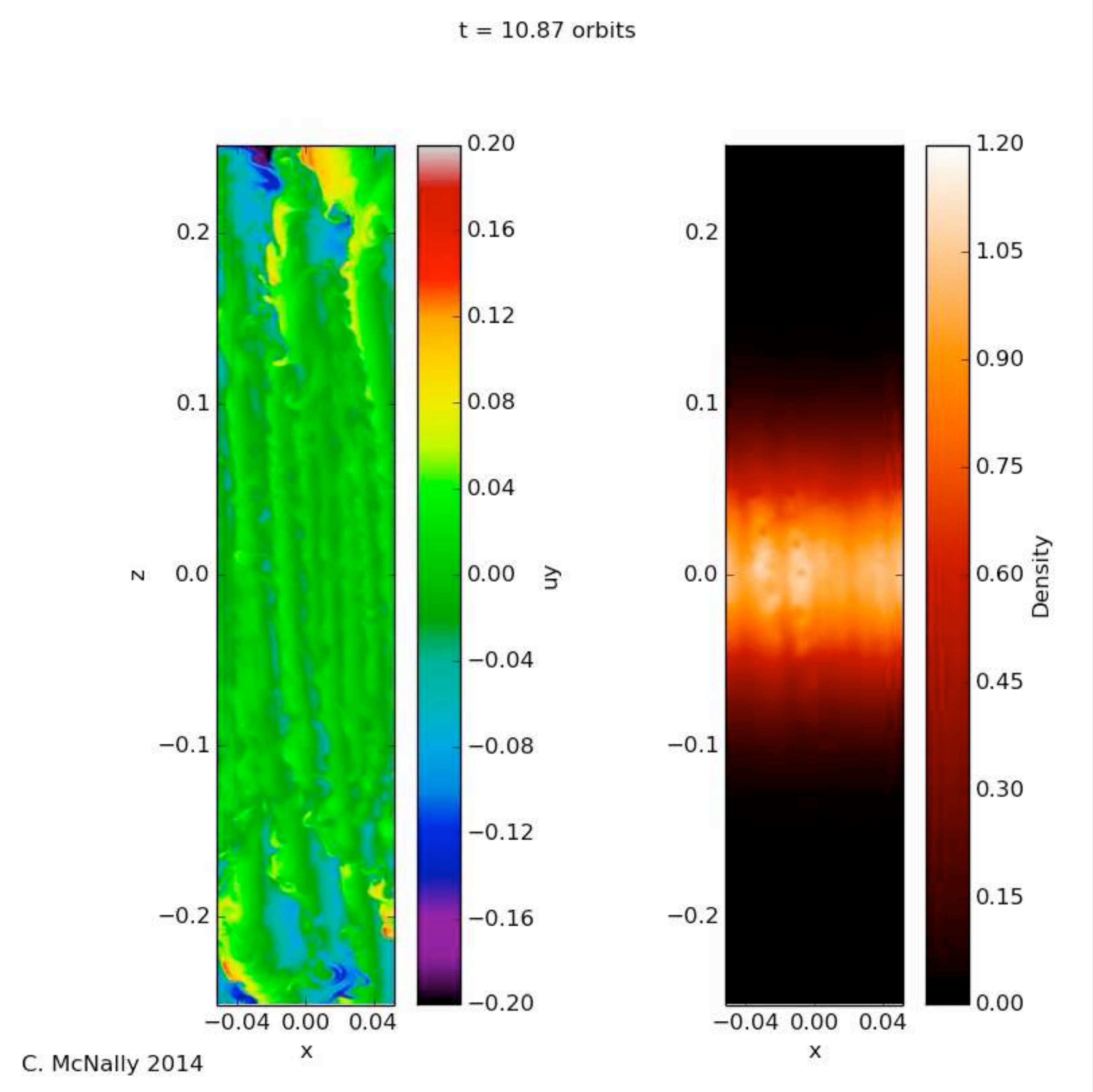
# Hydrodynamic Disk Instabilities




Nelson, Gressel, & Umurhan 2013

# A First Implementation of the VGSB




$$q = -1.5 \qquad p = -1.0 \qquad c = 0.05\, v_{\mathrm{K}}$$